Estimator Design for a Subsonic Rocket Car (Soft Landing) Based on State-Dependent Delay Measurement

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Abstract—We design an estimator for the position of a rocket car. The scheme is based on a novel idea of delay measurements in a realistic scenario. Ultrasonic sensors are used to give the measurement of the delay. The delay depends implicitly on the state of the system. This gives rise to a time delay system with a state-dependent delay which is a challenging research avenue. The central theme is to use inversion of the delay model and to extract the state vector given the delay. The overall system is nonlinear because of the state-dependent delay. We design an asymptotic observer for estimating the position of the system using delay information. We call or refer to this new technique as Delay Injection. We show that the error dynamics involve a time-varying delay instead of a state-dependent delay and this makes the problem tractable. A brief outline of the nonlinear observation for the autonomous system is also given. Though rocket car is a particular motivating example, the approach presented in this article can be used for the on-line estimation of states of generic systems from state-dependent delays affecting the system dynamics. Simulation results are given at the end which demonstrate the effectiveness, validity and usefulness of the proposed scheme.

Index Terms—Estimator design, state-dependent delay, delay measurements, rocket car, asymptotic observer.

I. INTRODUCTION AND MOTIVATION

The analysis of systems with state-dependent time delays is one of the most challenging research horizons in system and control theory. There are still a lot of open problems in this area. These include but are not limited to solving Cauchy problem associated with state-dependent delay differential equations, information structure development, statespace characterization and stability analysis. Some quite recent work in this field starts with toy systems as discussed in [2] and [3]. Besides the rocket car problem under consideration, these state-dependent delays also arise as retarded potentials in electromagnetic theory, congestion control in communication networks, biological models, population dynamics, automatic milling machines, extruders and neural networks to mention just a few.

The rocket car model also called double integrator, pure moment of inertia, space vehicle or soft landing is used as a benchmark problem is system and control theory. It is a second order system with no friction or damping term. The dynamics are simply governed by Newton’s second law of motion.

Consider the rocket car shown in Fig. 1. Let the instantaneous position of the rocket car w.r.t. the wall be denoted by $x_1(t)$ and its velocity be $x_2(t)$. Let the thrust of the engine be denoted by $u(t)$. Now, assuming unit mass, it follows from Newtonian dynamics that the rocket car can be modelled as,

$$x_1(t) = x_2(t)$$
$$x_2(t) = u(t)$$

It will be assumed that the control effort $u(.)$ is such that the rocket remains subsonic.

Now the transmitter $Tx$ of the ultrasonic sensor transmits a signal/pulse, it travels with the speed of sound $v$, it is echoed by the wall and is detected by the receiver $Rx$ of the sensor. The delay or the lag experienced by the pulse will give us the information of the position of the cart. Suppose the delay is denoted by $\tau$ and the rocket car had the position $x_1(t)$ at the moment when the pulse was received by the receiver $Rx$. This means that the position of the rocket car at the moment when the sound signal was transmitted was $x_1(t - \tau)$. Thus, the delay from transmission to reception is precisely given by,

$$\tau = \frac{x_1(t) + x_1(t - \tau)}{v}$$

In practice, one would have sampled delay observations, but for the sake of a simpler first approach (and “proof of concept”) we consider the observation as continuous.

Now we see from the above equation that the delay is a function of the state (position of the rocket car). Yes, it is of...
course a state-dependent delay. We also notice that there is an implicit dependence of the delay on the state. This makes the problem hard and challenging because position cannot be easily recovered from the delay measurement. We cannot use algebraic and mathematical machinery to extract \( x_1(t) \) from \( \tau \). This fact motivated us to estimate the position from delay measurement or observation which leads to the design of observer.

Practically, the speed of sound is quite fast compared to the distances involved in "the particular" application invoked by the authors of this paper. Hence the delayed state is very close to the real state and that is why the average is taken typically in practice. Nevertheless, it is always good to have a provably correct method rather than an approximation.

It is also emphasized here that the same technique can be used for position estimation of sonars using delay observation. In subsonic landing problem with sonar, it will not work unless there is a sound transmitting medium. In such a medium there is friction, so Newton’s equations as shown are inconsistent with the problem. It can also be used in robotics to avoid hitting the obstacles and in underwater vehicles to achieve soft “landing” on ocean floor. In the first case the dynamics are governed by the same equations as those for the rocket car whereas in the second case viscous friction models are pertinent.

II. LITERATURE SURVEY

Time delays systems in general and State-Dependent Delay Differential Equations (SDDE) in particular have always remained enigmatic and bizarre in systems theory literature. A proper system theoretic and rigorous treatment of time delay systems starts from the leading manuscript by Jack Hale [1]. In this book, the authors discuss systems with constant and time-varying delays. They make it clear that one should consider the state space as the function space consisting of the initial data on a delay interval and then consider such equations as evolutionary equations in this function space.

Some preliminary results on the asymptotic stability of systems with state-dependent delay were given in [6]. Lyapunov-Krasovskii (LK) theory is used to obtain stability conditions in the local framework. Hartung et al. in [12] illustrate the theory and applications of functional differential equations involving state-dependent delays, with emphasis on particular models and on the emerging theory from the dynamical systems point of view. In [10], the authors consider the effect of state-dependent delay on a weakly damped nonlinear oscillator. They also consider Hopf Bifurcation and persistent oscillations associated with SDDE.

Sipahi et al. in a featured article [18] discuss a wide spectrum of applications of time delay systems ranging from microscopic vehicular traffic flow, variable-pitch milling dynamics, biochemical feedback in cell regulatory networks and epidemics (hematology dynamics) to operations research. They show that delays are not always detrimental and can be a blessing in some cases. Delays can be used for the stabilization of unstable periodic orbits in systems involving chaos.

In [11], Adimy et al. model the dynamics of a dynamics of the population of Hematopoietic Stem Cells (HSCs) by a nonlinear SDDE. They use the method of characteristics to reduce a set of coupled partial differential equations to a nonlinear SDDE. Lyapunov-Razumikhin (LR) function based approach is used for the global stability analysis of the trivial steady state of the system. They also show the existence of Hopf bifurcation and stability switches. In [13], Walther models soft landing by an SDDE and it is shown that soft landing occurs for an open set of initial data, which is determined by means of a smooth invariant manifold.

Integrator back-stepping technique is used for forward-complete nonlinear systems involving state-dependent delays [15]. Delay compensation methods for nonlinear, adaptive, and Partial Differential Equations (PDE) systems are discussed in the monograph [14]. A compensation technique known as predictor-feedback design is used for the system characterized by an SDDE. Regional stability results and an estimate of the domain of attraction is given. In [16], the authors investigate a compensation technique for compensating state-dependent input delays for both linear and nonlinear systems. For nonlinear systems with state-dependent input delay which are, in the absence of the input delay, either forward complete and globally stabilizable or just locally stabilizable (by a possible time-varying control law), the authors of [16] and [17] design a predictor-based compensator. Again local results are obtained with a prescribed region of attraction.

This paper solves the problem of estimating the state based on time of arrival measurements. To the best of the knowledge of the authors of this paper, no work is done for the observation or estimation of systems involving state-dependent delays. There is also very little work done for implicit state-dependent delay systems. Output Injection technique is ubiquitous in systems and control theory for observer design. The technique used in this paper, we call it Delay Injection, is used for the first time for state estimation.

III. ESTIMATION BASED ON DELAY

We express the dynamics of the system in a general vector matrix form as follows,

\[
\dot{x}(t) = Ax(t) + Bu(t)
\] (4)

Here \( x(t) \in \mathbb{R}^n \) is the state vector and \( u(t) \in \mathbb{R}^p \) is the control input. Now,

\[
x_1(t) = Cx(t)
\] (5)

and therefore,

\[
\tau = \frac{Cx(t) + Cx(t - \tau)}{v}
\] (6)

Denoting \( \frac{C}{v} \) by \( M \), we have,

\[
\tau = Mx(t) + Mx(t - \tau)
\] (7)
We measure the delay \( \tau(t) \) at each instant \( t \) with help of ultrasonic sensors and construct the observer for the state \( x(t) \) as follows:

\[
\dot{x}(t) = Ax + Bu(t) + L(\tau - \hat{\tau})
\]  
(8)

where the last term is the correction term with \( L \) being the gain of the estimator and \( \hat{\tau} \) is precisely given by,

\[
\hat{\tau} = M\dot{x}(t) + Mx(t - \tau)
\]  
(9)

Substitution for \( \tau \) and \( \hat{\tau} \) in (8) yields,

\[
\dot{x}(t) = A\dot{x}(t) + Bu(t) + LM(e(t) - e(t - \tau(t)))
\]  
(10)

where \( e(t) = x(t) - \dot{x}(t) \) is the estimation error at time \( t \).

Now we subtract the above equation from the original dynamic equation of the system as given by (4) to get the error dynamics for the estimator as follows,

\[
\dot{e}(t) = (A - LM)e(t) - LM(e(t) - e(t - \tau(t)))
\]  
(11)

From the above equation it is obvious that the error dynamics satisfy a functional differential equation involving a time-varying delay. Now, the problem is tractable. The only design parameter is the matrix \( L \). The error will converge asymptotically to zero if the above delay differential equation is asymptotically stable or in other words the origin \( e(t) = 0 \) is asymptotically stable.

IV. CAUSALITY CONSTRAINT

After the pioneering work on well-posedness of systems with time varying delays [4] and [5], it is now very well known that \( \hat{\tau}(t) \leq 1 \) is not just a technical condition to make the analysis tractable and proof work but rather it is a causality constraint. Violation of this constraint will make the system non causal and ill-posed. Non-causal systems are practically not realizable and do not make sense physically. We now analyze this constraint and show that it is in perfect agreement with the physical reality in the problem under consideration. From (3), we have

\[
\hat{\tau}(t) = \frac{\dot{x}_1(t) - (1 - \hat{\tau})x_1(t - \tau(t))}{v} \\
\dot{\hat{\tau}}(t) = \frac{\dot{x}_1(t) - (1 - \hat{\tau})x_1(t - \tau(t))}{v - \dot{x}_1(t - \tau(t))} \\
\hat{\tau}(t)v - \dot{x}_1(t - \tau(t)) = x_1(t) - \dot{x}_1(t - \tau(t)) \\
\hat{\tau}(t) = \frac{x_1(t) - \dot{x}_1(t - \tau(t))}{v - \dot{x}_1(t - \tau(t))}
\]  
(12)

Now, \( \hat{\tau}(t) \leq 1 \) translates precisely to \( x_1(t) \leq v \) which is quite natural and has a reasonable physical meaning. How can the ultrasonic sensor measure the position of the rocket car if the speed of the rocket exceeds the speed of sound?

V. STABILITY ANALYSIS OF THE ERROR DYNAMICS

For the estimator to work, the asymptotic stability of the error dynamics equation (11) is of prime importance. Unless (11) is asymptotically stable, there is no question of estimator gain \( L \) design. Since (11) is a time varying delay differential equation, we consider the linear system with a time-varying delay as characterized by the following form of retarded functional differential equation or differential-difference equation of the form

\[
\dot{x}(t) = A_0x(t) + A_1x(t - \tau(t)) \quad \forall t \geq 0 \\
x(t) = \psi(t). \quad \forall t \in [-h, 0]
\]  
(13)

Where \( x(t) \in \mathbb{R}^n \) is the state variable, \( \psi(t) \in \mathcal{C}([-h, 0], \mathbb{R}^n) \) is the initial infinite dimensional history function living in the Banach function space. Here \( \mathcal{C}([-h, 0], \mathbb{R}^n) \) denotes the Banach space of continuous vector functions mapping the interval \([-h, 0]\) to \( \mathbb{R}^n \) with the topology of uniform convergence. The matrices \( A_0 \) and \( A_1 \) each have size \( n \times n \) and the time-varying delay \( \tau(t) \) and its rate are characterized as follows.

\[
\tau(t) \leq h. \quad \hat{\tau}(t) \leq d \leq 1
\]  
(14)

i.e. \( h \) and \( d \) are the upper bounds on the size of the time-varying delay \( \tau(t) \) and its rate respectively. We give two ways for the stability analysis. The first technique is based on Riccati equation and the second one is based on LMIs.

A. Robust Stability Based on Riccati Equation

We directly use the following corollary 1 as derived in [7].

**Corollary 1:** If there exists a triple of time-invariant positive definite matrices \( P, Q, R \) together with a scalar constant \( \alpha \geq 1 \) such that

\[
A_0^T P + PA + Q + \alpha^2 PA \alpha^{-1} A_1^T P = 0
\]  
(15)

then the system (13) is robustly \( H_\infty \)-asymptotically stable, where \( H_\infty = \{ \tau(.) | 0 \leq \tau(.) \leq h, \ \hat{\tau}(.) \leq 1 - \frac{1}{\alpha^2} \} \).

Replacing \( A_0 \) by \( A - LM \) and \( A_1 \) by \( -LM \) in the above Riccati equation (15), our estimator gain \( L \) must satisfy

\[
(A - LM)^T P + P(A - LM) + Q + R + \alpha^2 PLMQ^{-1}M^T L^T P = 0
\]  
(16)

**Remark:**

Notice that the results given by the Riccati equation (16) are highly conservative. No feasible solution was found using this equation. A full-fledged time-varying case for the matrices \( P(t), Q(t) \) and \( R(t) \) may give less restrictive results but was not attempted. However, this conservatism is reduced by using the linear matrix inequality based approach in the next subsection which has more flexibility and extra degrees of freedom of the matrix variables involved.

B. Stability Bounds Analysis Using LMIs

An auxiliary dynamical system [9] or a comparison system is introduced here as given below.

\[
\dot{y}(t) = A_2y(t) + A_3x(t), \quad y(t) \in \mathbb{R}^n
\]  
(17)

We construct the following augmented system which comprises (17) and the original delay-differential equation i.e. the top equation in (13).
\[
\begin{bmatrix}
    \dot{x}(t) \\
    \dot{y}(t)
\end{bmatrix} = \begin{bmatrix}
    A_0 & 0 \\
    A_3 & A_2
\end{bmatrix} \begin{bmatrix}
    x(t) \\
    y(t)
\end{bmatrix} + \begin{bmatrix}
    A_1 & 0 \\
    0 & 0
\end{bmatrix} \begin{bmatrix}
    x(t - \tau(t)) \\
    y(t - \tau(t))
\end{bmatrix}
\]

(18)

Here, we bring in \( e(t) \) such that
\[
e(t) = x(t - h) - y(t)
= x(t - h) - \int_{t-\tau(t)}^{t} \dot{y}(t)dt - y(t - \tau(t))
= x(t - h) - \int_{t-\tau(t)}^{t} (A_2 y(\beta) + A_3 x(\beta))d\beta - y(t - \tau(t))
\]

(19)

Then system (13) can be rewritten as
\[
\begin{align*}
    \dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau(t)) + A_1 y(t) - A_1 y(t) \\
    &= A_0 x(t) + A_1 e(t) + A_1 y(t) \\
    &= A_0 x(t) + A_1 y(t) + A_1 x(t - \tau(t)) - A_1 \\
    &\quad \times \int_{t-\tau(t)}^{t} (A_2 y(\beta) + A_3 x(\beta))d\beta - A_1 y(t - \tau(t))
\end{align*}
\]

(20)

From (17) and (20), the following augmented system is obtained.
\[
\begin{bmatrix}
    \dot{x}(t) \\
    \dot{y}(t)
\end{bmatrix} = \begin{bmatrix}
    A_0 & A_1 \\
    A_3 & A_2
\end{bmatrix} \begin{bmatrix}
    x(t) \\
    y(t)
\end{bmatrix}
+ \begin{bmatrix}
    A_1 & -A_1 \\
    0 & 0
\end{bmatrix} \begin{bmatrix}
    x(t - \tau(t)) \\
    y(t - \tau(t))
\end{bmatrix}
+ \begin{bmatrix}
    A_1 \int_{t-\tau(t)}^{t} (A_2 y(\beta) + A_3 x(\beta))d\beta \\
    0
\end{bmatrix}
\]

(21)

Now, we re-organize the coupled system (21) to take the form given below.
\[
\dot{\phi}(t) = \hat{A}_0 \phi(t) + \hat{A}_1 \phi(t - \tau(t)) + \phi(t)
\]

(22)

where
\[
\hat{A}_0 = \begin{bmatrix}
    A_0 & A_1 \\
    A_3 & A_2
\end{bmatrix}, \hat{A}_1 = \begin{bmatrix}
    A_1 & -A_1 \\
    0 & 0
\end{bmatrix},
\]

\[
\phi(t) = \begin{bmatrix}
    A_1 \int_{t-\tau(t)}^{t} (A_2 y(\beta) + A_3 x(\beta))d\beta \\
    0
\end{bmatrix},
\]

\[
\phi(t) = \begin{bmatrix}
    x(t) \\
    y(t)
\end{bmatrix}
\]

Finally, we invoke the following theorem to obtain a sufficient condition.

**Theorem 1:** The time-delay system (13) is uniformly asymptotically stable if there exists a matrix \( N \in \mathbb{R}^{n \times 2n} \), \( 2n \times 2n \) symmetric positive definite matrices \( R, S \) and an \( n \times n \) symmetric positive definite matrix \( Z \), such that the following LMI holds:
\[
\Omega := \begin{bmatrix}
    \Omega_{11} & * & * \\
    \Omega_{21} & \Omega_{22} & * \\
    \Omega_{31} & \Omega_{32} & \Omega_{33}
\end{bmatrix} < 0
\]

(23)

where an ellipsis * represents an easily induced symmetric block, and
\[
\begin{align*}
    \Omega_{11} &= \begin{bmatrix}
    A_0 & A_1 \\
    0 & 0
\end{bmatrix} R + R \begin{bmatrix}
    A_0 & A_1 \\
    0 & 0
\end{bmatrix}^T + \begin{bmatrix}
    0 & 0 \\
    N & 0
\end{bmatrix} \\
    + \begin{bmatrix}
    0 & N^T \\
    0 & 0
\end{bmatrix} + S + h \begin{bmatrix}
    A_1 & 0 \\
    0 & 0
\end{bmatrix} Z \begin{bmatrix}
    A_1^T & 0 \\
    0 & -A_1^T
\end{bmatrix}
\end{align*}
\]

(24)

Using the time derivative of this LKF functional along the trajectory of the system, applying the matrix substitutions \( R = P^{-1}, S = RQR \) and \( N = \begin{bmatrix}
    A_3 & A_2 \\
\end{bmatrix} R \) and Shur complements, the LMI is formulated. The detailed proof is accomplished in [8].

Again replacing \( A_0 \) by \( A - LM \) and \( A_1 \) by \( -LM \) in the above LMI of Theorem 1 and applying congruence transformation, the estimator gain \( L \) can be computed.

**Comment:**

Notice that the estimator should be designed such that it is faster than the speed of the cart, otherwise the rocket car may hit the wall.

**VI. NONLINEAR OBSERVATION FOR AUTONOMOUS VERSION OF THE SYSTEM**

Notice that application of Taylor series expansion to the right hand side of (3) yields,
\[
v_\tau(t) = x_1(t) + x_1(t - \tau(t)) \dot{x}_1(t) + \frac{\tau^2(t)}{2!} \ddot{x}_1(t) - \ldots \ldots
\]

(25)

For the autonomous version of the system \( u(t) = 0 \) and therefore \( \dot{x}_1(t) = x_2(t) = 0 \) and all the subsequent higher derivatives of \( x_1(t) \) in the above expansion are zero. This yields the following exact expression (no truncations),
\[ \tau(t) = \frac{1}{v}(2x_1(t) - \tau(t)\dot{x}_1(t)) \]  

(29)

Finally, we achieve an explicit expression for the delay \( \tau(t) \) in terms of the state variables \( x_1(t) \) and \( x_2(t) \) as,

\[ \tau(t) = \frac{2x_1(t)}{x_2(t) + v} \]  

(30)

By analogy with the extended Kalman filter design, one may be tempted to use an estimate of \( \tau(t) \) as follows,

\[ \hat{\tau}(t) = \frac{2\hat{x}_1(t)}{x_2(t) + v} \]  

(31)

Now for the above \( \hat{\tau}(t) \) the the nonlinear observer equations with delay injection are given as follows,

\[ \dot{\hat{x}}_1(t) = \dot{\hat{x}}_2(t) + l_1(\tau(t) - \hat{\tau}(t)) \]  

(32)

\[ \dot{\hat{x}}_2(t) = l_2(\tau(t) - \hat{\tau}(t)) \]  

(33)

where \( l_1 \) and \( l_2 \) are the gains of the nonlinear observer. However, our simulation results reveal that the error does not decay to zero asymptotically unlike the linear asymptotic observer (9). This is because of the nonlinear nature of the estimator dynamics. We also used higher derivatives of \( \tau(t) \) in our observation equations to get instantaneous observability conditions. In the ideal (pure mathematical) case, exact knowledge of \( \tau \) implies knowledge of \( \hat{\tau} \), as used in Kalman’s original observability criterion.

\[ \hat{\tau}(t) = \frac{2x_2(t)}{x_2(t) + v} \]  

(34)

From this equation, the inversion equations for \( x_1(t) \) and \( x_2(t) \) in terms of \( \tau \) and \( \hat{\tau} \) are given by,

\[ x_1(t) = \frac{\tau v}{2 - \hat{\tau}} \]  

(35)

\[ x_2(t) = \frac{\hat{\tau} v}{2 - \hat{\tau}} \]  

(36)

We observe here that \( \hat{\tau}(t) \neq 2 \) is the observability condition. But this is already fulfilled by the strong causality constraint.

In order to extend the above theory to closed loop systems, we use the concept of “borrowed state-feedback”. Assume that a stabilizing feedback exists, then the closed loop is autonomous, hence the inversion can be done exactly. We do this inversion to obtain \( \dot{x} \), (we call it the “inverted state”) for the closed loop system. Now we break the feedback open and insert the inverted state where we used \( x \) before. This puts the observer after the controller, instead of first observing then controlling, but in LTI, this is commutative anyway. It is a simple paradigm shift. This is because the deterministic separation principle implies that the observer and controller design are completely independent in LTI systems. However, no such separation exists for nonlinear dynamical systems.

With the designed controller \( u(t) = -k_1\dot{x}_1(t) - k_2\dot{x}_2(t) \), the closed loop system is autonomous, i.e. \( \dot{x}_1(t) = \dot{x}_2(t) = u(t) = -k_1\dot{x}_1(t) - k_2\dot{x}_2(t) \). By an appropriate choice of the static state-feedback gains \( k_1 \) and \( k_2 \), the transient behavior (overshoot and settling time or speed of response) of the rocket car can be controlled. This is the so-called regulator design problem. Practically speaking, the rocket car should move with uniform acceleration to avoid jerks in the motion. Therefore, it is assumed that all the derivatives of \( u(t) \) are zero and an exact expression like (30) holds. Once the regulator design is accomplished, the nonlinear estimator will give the estimated states \( \hat{x}_1 \) and \( \hat{x}_2 \). Now, the control law will be practically implemented as \( u(t) = -k_1\dot{x}_1(t) - k_2\dot{x}_2(t) \).

Further on this is part of the future research.

VII. Simulation Results

In this example, we design an asymptotic observer for the autonomous system. Clearly, for the rocket car discussed here, \( \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Fig. 2 shows the original position of the rocket car and its estimate. The initial conditions for the system are \( \begin{pmatrix} 7.5 \\ -0.5 \end{pmatrix} \) and those for the estimator are \( \begin{pmatrix} 1.5 \\ -0.3 \end{pmatrix} \). The gain of the asymptotic estimator was chosen as \( \mathbf{L} = \begin{pmatrix} 90 \\ 100 \end{pmatrix} \). The speed of sound \( v \) in the air was taken as 332 m/s and the delay \( \tau(t) = \frac{15-t}{0.5} \) in a realistic fashion. A set of positive definite and symmetric matrices \( \mathbf{R}, \mathbf{S} \) and \( \mathbf{Z} \) which makes the inequality (23) feasible were found using Matlab and are given as follows.

\[
\mathbf{R} = \begin{pmatrix}
59.2052 & 12.4865 & 37.5752 & 0 \\
12.4865 & 39.1904 & 25.2265 & 0 \\
37.5752 & 25.2265 & 46.2072 & 0 \\
0 & 0 & 0 & 32.9406 \\
\end{pmatrix}
\]

\[
\mathbf{S} = \begin{pmatrix}
17.0862 & -2.2315 & -1.8066 & 0 \\
-2.2315 & 10.9154 & 7.1515 & 0 \\
-1.8066 & 7.1515 & 26.6621 & 0 \\
0 & 0 & 0 & 23.7484 \\
\end{pmatrix}
\]

\[
\mathbf{Z} = \begin{pmatrix}
4.1265 & 0 & 0 \\
0 & 2.9230 \\
\end{pmatrix}
\]

The rectangular matrix \( \mathbf{N} \) in (23) was found as follows.

\[
\mathbf{N} = \begin{pmatrix}
-0.7365 & 20.4172 & -25.8251 & 0 \\
0 & 0 & 0 & -25.8743 \\
\end{pmatrix}
\]

Fig. 3 shows the profiles for position error \( e_1(t) \) and velocity error \( e_2(t) \) of the rocket car. We see that the error decays down to zero as time progresses. In other words the estimated position approaches the actual position. Same is the case for the velocity.

VIII. Concluding Remarks

A. Conclusions

In precise and concise words, the main innovation in this paper is the inversion of the state-dependent delay model (3). This can either be accomplished exactly using Taylor series expansion or in asymptotic fashion. The former inversion technique only works for autonomous case. The objective in both the cases is the retrieval or recovery of the state vector using delay information. Here we presented an estimator.
A design approach for measuring the position of a subsonic rocket car using observations based on delay measurement. The delay in the original problem was state dependent. Nevertheless, we showed that the error dynamics of the estimator contain a time-varying delay which makes the estimation problem tractable. An asymptotic observer was designed. The simulation results bring to light the effectiveness of the estimator. Causality constraints required for making the problem well-posed were determined. These constraints are in excellent agreement with the physical reality. A novel estimation technique called Delay Injection was introduced. Nonlinear observation technique was also established for the autonomous version of the system.

B. Future Works

Future tasks include formulation of instantaneous observability conditions for the nonlinear observation, consideration of Doppler effect, optimal estimation and incorporation of stochastics to account for measurement noise.

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